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DIRICHLET PROBLEM AT INFINITY FOR HARMONIC MAPS: RANK ONE SYMMETRIC SPACES

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ABSTRACT. Given a symmetric space M, of rank one and noncompact type, one compactifies M by adding a sphere at infinity, to obtain a manifold M' with boundary. If $\overline{M}$ is another rank one symmetric space, suppose that $f: \partial M' \to \partial \overline{M}'$ is a continuous map. The Dirichlet problem at infinity is to construct a proper harmonic map $u: M \to \overline{M}$ with boundary values f. This paper concerns existence, uniqueness, and boundary regularity for this Dirichlet problem.

1. INTRODUCTION

Let M and $\overline{M}$ be complete simply connected manifolds of strictly negative curvature. One may compactify M and $\overline{M}$, using asymptotic classes of geodesic rays, by adding spheres at infinity. We denote the compactifications by M' and $\overline{M}'$, and the spheres added at infinity by $\partial M'$, $\partial\overline{M}'$. Suppose that $f: \partial M' \to \partial\overline{M}'$ is a continuous map. The Dirichlet problem at infinity consists of finding a harmonic map $u: M \to \overline{M}$, with boundary values f at infinity. Here one means that $u \in C^2(M, \overline{M}) \cap C^0(M', \overline{M}')$, and the boundary values f are taken continuously. In general, the Dirichlet problem at infinity seems to be quite difficult. If M and $\overline{M}$ both have constant negative curvature, then Li and Tam [8, 9] have proved a number of significant results, concerning uniqueness, existence, and boundary regularity. Our plan is to extend this discussion to the context of rank one symmetric spaces.

Suppose now that M and $\overline{M}$ are rank one symmetric spaces of noncompact type. In the unbounded model, M is realized as $R^+ \times N$, where R^+ is the positive real line and N is a two term nilpotent group. The Lie algebra of N decomposes as $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_2$ is central in $\mathfrak{n}$ and $[\mathfrak{n}_1, \mathfrak{n}_1] \subset \mathfrak{n}_2$. For the exceptional case, of constant negative curvature on M, we adopt the convention that $\mathfrak{n}_1$ is the entire abelian Lie algebra and $\mathfrak{n}_2$ is empty. In the unbounded model $R^+ \times N$, the metric of M is realized as a doubly warped product [1]:

$$g_M = \left(\frac{dy}{y}\right)^2 + y^{-2} g_{\mathfrak{n}_1} + y^{-4} g_{\mathfrak{n}_2}. \tag{1.1}$$

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Here $y \in R^+$ is the coordinate on the first factor of $R^+ \times N$. Via Cayley transform, this provides local coordinate charts at the boundary $\partial M'$ of the compactification. Of course, the same discussion applies to $\overline{M}'$. To formulate our results, we introduce indices $0 \le j \le n_1 + n_2$. The index 0 refers to $\partial/\partial y$, the indices $1 \le j \le n_1$ allude to the g_{n_1} part of (1.1); and the indices $n_1+1 \le j \le n_1+n_2$ refer to the g_{n_2} part of (1.1). On $\overline{M}$, we use corresponding Greek indices $0 \le \alpha \le \overline{n}_1 + \overline{n}_2$.

Our first observation is that if f is the boundary value of a harmonic map $u \in C^2(M, \overline{M}) \cap C^1(M', \overline{M}')$, then $f_j^\gamma = 0$, $1 \le j \le n_1$, $\overline{n}_1 + 1 \le \gamma \le \overline{n}_1 + \overline{n}_2$. Here f_j^γ are the components of the differential of f. By contrast, Li and Tam [8] proved that, for spaces of constant negative curvature, any f with nonvanishing energy density can occur as the boundary value of a $C^1(M', \overline{M}')$ harmonic map. If h is a harmonic self-map of the unit ball in C^n, with its Bergman metric, the condition $f_j^\gamma = 0$ means that f is a contact transformation.

We now describe our uniqueness results. Suppose that both h and $\hat{h}$ are proper harmonic maps between rank one symmetric space M and $\overline{M}$. Assume h, $\hat{h} \in C^2(M', \overline{M}')$ have the same boundary value $f: \partial M' \to \partial \overline{M}'$, which satisfies

$$\sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} f_j^\gamma f_j^\gamma > 0.$$

Then h and $\hat{h}$ are identical. If the range $\overline{M}$ has constant negative curvature, then one only needs h, $\hat{h} \in C^1(M', \overline{M}')$. If the common boundary value satisfies

$$\sum_{j=1}^{n_1} \sum_{\gamma=1}^{\overline{n}_1} f_j^\gamma f_j^\gamma > 0$$

then $h = \hat{h}$. This last result was proved by Li and Tam in [8] when both M and $\overline{M}$ have constant negative curvature, and their proof is similar to ours.

Our basic existence result assumes that one is given boundary data $f \in C^{2,\varepsilon}(\partial M', \partial \overline{M}')$, $0 < \varepsilon < 1$, so that $f_j^\gamma = 0$, $1 \le j \le n_1$, $\overline{n}_1 + 1 \le \gamma \le \overline{n}_1 + \overline{n}_2$, and $\sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} f_j^\gamma f_j^\gamma > 0$. We construct a harmonic map $u: M \to \overline{M}$, which assumes the boundary values f continuously. If the range $\overline{M}$ has constant negative curvature, it is enough to assume $f \in C^{1,\varepsilon}(\partial M', \partial \overline{M}')$, $0 < \varepsilon < 1$, and $\sum_{j=1}^{n_1} \sum_{\gamma=1}^{\overline{n}_1} f_j^\gamma f_j^\gamma > 0$. We prove the existence of a harmonic map u, which assumes the boundary values f continuously. If both the domain and range have constant negative curvature, this again reduces to a result of [8], where similar arguments were employed.

For our regularity results, we assume that $f \in C^{l+2,\varepsilon}(\partial M', \partial \overline{M}')$, $0 \le l < n_1 + 2n_2$, $0 < \varepsilon < 1$, satisfies $f_j^\gamma = 0$, $1 \le j \le n_1$, $\overline{n}_1 + 1 \le \gamma \le \overline{n}_1 + \overline{n}_2$, and $\sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} f_j^\gamma f_j^\gamma > 0$. We prove that there is a harmonic map, $u: M \to \overline{M}'$, $u \in C^{k+1,\overline{\varepsilon}}(M', \overline{M}') \cap C^2(M, \overline{M})$, with boundary values f, when $\overline{\varepsilon} < \varepsilon$ and $-2 \le 2k < l - 1$. Although the factor 2 in $2k$ is not appealing, it may be needed. In [4], Graham studied the model linear problem of the Bergman Laplacian on the unit ball in C^n, where similar difficulties appear. If both M

and $\overline{M}$ have constant negative curvature, then one instead assumes that $f \in C^{l+1,\varepsilon}(\partial M', \partial\overline{M}')$, $0 < \varepsilon < 1$, $0 \leq l < n_1$, and $\sum_{j=1}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} f_j^\gamma f_j^\gamma > 0$. Then there exists a harmonic map $u: M \to \overline{M}$, $u \in C^{l+1,\varepsilon}(M', \overline{M}') \cap C^2(M, \overline{M})$, with boundary values f. This was proved earlier in [8]. However, our method of proof is somewhat different. We replace the use of conformality to the Euclidean Laplacian with ideas based upon the facts that our spaces M, $\overline{M}$ admit transitive isometry groups.

2. Tension in an adapted frame field

Let $h: M \to \overline{M}$ be a C^2 map between Riemannian manifolds M and $\overline{M}$. The differential dh may be regarded as a section of $T^*M \otimes h^{-1}T\overline{M}$. The bundle $T^*M \otimes h^{-1}T\overline{M}$ admits a connection ∇ induced by the Levi-Civita connection on M and the pull-back of the Levi-Civita connection on $\overline{M}$. One defines the tension $\tau(h) = \operatorname{Tr}(\nabla dh)$. For later reference, we compute $\tau(h)$ explicitly, especially in the case where M and $\overline{M}$ are rank one symmetric spaces of noncompact type.

Suppose that e_i is a local frame field on TM, with dual coframe field e_i^*, $i = 1, 2, \dots, \dim M$. Let f_α be a local frame field for $T\overline{M}$, $\alpha = 1, 2, \dots, \dim\overline{M}$. We do not assume that either e_i or f_α is orthonormal. We may denote the differential of h as $dh = h_i^\alpha e_i^* \otimes f_\alpha$, where one sums over both the indices i and α. Let $g^{ij} = \langle e_i^*, e_j^*\rangle$ and $h_{ij}^\alpha = e_j h_i^\alpha$. Calculating from the definition yields

$$\begin{aligned}\tau(h) &= \operatorname{Tr}(\nabla dh) = \operatorname{Tr}(e_j^* \otimes \nabla_{e_j}(h_i^\alpha e_i^* \otimes f_\alpha)) \\ &= g^{ij}h_{ij}^\alpha f_\alpha + h_i^\alpha \langle e_j^*, \nabla_{e_j} e_i^*\rangle f_\alpha + h_i^\alpha g^{ij}\nabla_{e_j} f_\alpha\end{aligned}$$

where summation is understood over i, j, and α.

We rewrite the formula for $\tau(h)$ by noting that

$$\begin{aligned}\langle e_j^*, \nabla_{e_j} e_i^*\rangle &= g^{jk}e_k(\nabla_{e_j}e_i^*) = -g^{jk}e_i^*(\nabla_{e_j}e_k), \\ \nabla_{e_j} f_\alpha &= \nabla_{h_j^\beta f_\beta} f_\alpha = h_j^\beta \nabla_{f_\beta} f_\alpha = h_j^\beta f_\gamma^*(\nabla_{f_\beta} f_\alpha) f_\gamma .\end{aligned}$$

Thus $\tau(h) = \tau^\alpha(h) f_\alpha$, where

$$\tau^\alpha(h) = g^{ij}h_{ij}^\alpha - g^{jk}e_i^*(\nabla_{e_j}e_k)h_i^\alpha + g^{ij}h_i^\gamma h_j^\beta f_\alpha^*(\nabla_{f_\beta} f_\gamma). \tag{2.1}$$

The summation convention is employed for the indices i, j, k, β, and γ.

The next step is to give more explicit expressions for $\tau(h)$ in the special case where M on $\overline{M}$ are both rank one symmetric spaces of noncompact type. In the unbounded model, M is realized as $R^+ \times N$, where R^+ is the positive real line and N is a two term nilpotent Lie group. The Lie algebra of N decomposes as $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_2$ is the central in $\mathfrak{n}$ and $[\mathfrak{n}_1, \mathfrak{n}_1] \subset \mathfrak{n}_2$. The hyperbolic space of constant negative curvature is exceptional, and N reduces to the abelian group $R^{\dim M - 1}$. For the hyperbolic space, we adopt the convention that $\mathfrak{n}_1$ is the entire abelian Lie algebra and $\mathfrak{n}_2$ is empty. Choose an orthonormal basis $X_1, X_2, \dots, X_{n_1}$ for $\mathfrak{n}_1$ and $Z_1, Z_2, \dots, Z_{n_2}$ for $\mathfrak{n}_2$, relative to a left invariant metric on N. Here $n_1 = \dim \mathfrak{n}_1$, $n_2 = \dim \mathfrak{n}_2$, and thus $\dim M = n_1 + n_2 + 1$. One has $[X_i, Z_j] = [Z_j, Z_k] = 0$ and $[X_i, X_j] = a_{ij}^k Z_k$, for some structure constants a_{ij}^k. A sum is understood over k. In the unbounded model $R^+ \times N$, the metric of M is realized as a doubly warped product [1]:

$$g_M = \left(\frac{dy}{y}\right)^2 \oplus y^{-2} g_{\mathfrak{n}_1} \oplus y^{-4} g_{\mathfrak{n}_2}, \qquad y > 0. \tag{2.2}$$

Here $y \in R^+$ is the coordinate on the first factor of $R^+ \times N$. Moreover, $g_{\mathfrak{n}_1} + g_{\mathfrak{n}_2}$ is a left invariant metric on N. Of course, the same discussion applies to $\overline{M}$, where we denote the corresponding quantities with a bar, for example $\overline{X}_i$ are an orthonormal basis of $\bar{\mathfrak{n}}_1$.

On any Riemannian manifold, with metric g, there is a standard elementary formula [6] for the Levi-Civita connection:

$$\begin{aligned} 2g(A, \nabla_C B) = {} & Cg(A, B) + Bg(A, C) - Ag(B, C) \\ & + g(C, [A, B]) + g(B, [A, C]) - g(A, [B, C]) \end{aligned}$$

where A, B, C are vector fields. Using this formula, a lengthy but straightforward computation gives the connection ∇, in the frame field $\partial/\partial y$, X_i, Z_j, of M:

$$\begin{aligned} \nabla_{\partial/\partial y} \frac{\partial}{\partial y} &= -y^{-1} \frac{\partial}{\partial y}, \\ \nabla_{X_i} \frac{\partial}{\partial y} &= \nabla_{\partial/\partial y} X_i = -y^{-1} X_i, \\ \nabla_{Z_i} \frac{\partial}{\partial y} &= \nabla_{\partial/\partial y} Z_i = -2y^{-1} Z_i, \\ \nabla_{X_i} X_j &= y^{-1} \delta_{ij} \frac{\partial}{\partial y} + \frac{1}{2} a_{ij}^k Z_k, \\ \nabla_{Z_i} Z_j &= 2y^{-3} \delta_{ij} \frac{\partial}{\partial y}, \\ \nabla_{X_i} Z_j &= \nabla_{Z_j} X_i = \frac{1}{2} y^{-2} a_{ki}^j X_k. \end{aligned} \tag{2.3}$$

In the exceptional case where M is the hyperbolic space, there are no Z_i's, and (2.3) becomes

$$\begin{aligned} \nabla_{\partial/\partial y} \frac{\partial}{\partial y} &= -y^{-1} \frac{\partial}{\partial y}, \\ \nabla_{X_i} \frac{\partial}{\partial y} &= \nabla_{\partial/\partial y} X_i = -y^{-1} X_i, \\ \nabla_{X_i} X_j &= y^{-1} \delta_{ij} \frac{\partial}{\partial y}. \end{aligned} \tag{2.3a}$$

Of course, the frame field $\partial/\partial y$, X_i, Z_j is orthogonal but not orthonormal for the metric g_M. Sometimes, it will be useful to employ the orthonormal frame field $y\partial/\partial y$, yX_i, y^2Z_j, where one has the corresponding expressions

$$\begin{aligned}
&\nabla_{y\partial/\partial y}\left(y\frac{\partial}{\partial y}\right) = 0,\\
&\nabla_{yX_i}\left(y\frac{\partial}{\partial y}\right) - \nabla_{y\partial/\partial y}(yX_i) = \left[yX_i, y\frac{\partial}{\partial y}\right] = -yX_i,\\
&\nabla_{y\partial/\partial y}(yX_i) = 0,\\
&\nabla_{y^2Z_i}\left(y\frac{\partial}{\partial y}\right) - \nabla_{y\partial/\partial y}(y^2Z_i) = \left[y^2Z_i, y\frac{\partial}{\partial y}\right] = -2y^2Z_i,\\
&\nabla_{y\partial/\partial y}(y^2Z_i) = 0,\\
&\nabla_{yX_i}(yX_j) = \delta_{ij}y\frac{\partial}{\partial y} + \frac{1}{2}a_{ij}^k y^2 Z_k,\\
&\nabla_{y^2Z_i}(y^2Z_j) = 2\delta_{ij}y\frac{\partial}{\partial y},\\
&\nabla_{yX_i}(y^2Z_j) = \nabla_{y^2Z_j}(yX_i) = \frac{1}{2}a_{ki}^j yX_k.
\end{aligned} \tag{2.4}$$

The advantage of the orthonormal frame field $y\partial/\partial y$, yX_i, y^2Z_j lies in fact that the coefficients, on the right-hand side of (2.4), are independent of y. Also, for the hyperbolic space, (2.4) becomes

$$\begin{aligned}
&\nabla_{y\partial/\partial y}\left(y\frac{\partial}{\partial y}\right) = 0,\\
&\nabla_{yX_i}\left(y\frac{\partial}{\partial y}\right) - \nabla_{y\partial/\partial y}(yX_i) = \left[yX_i, y\frac{\partial}{\partial y}\right] = -yX_i,\\
&\nabla_{y\partial/\partial y}(yX_i) = 0,\\
&\nabla_{yX_i}(yX_j) = \delta_{ij}y\frac{\partial}{\partial y}.
\end{aligned} \tag{2.4a}$$

Returning to the local expression for the tension field, we choose the frame field e_i on M to consist of $e_0 = \partial/\partial y$; $e_i = X_i$, $1 \le i \le n_1$; $e_i = Z_{i-n_1}$, $n_1 + 1 \le i \le n_1 + n_2$. Similarly, on $\overline{M}$ it is natural to select $f_0 = \partial/\partial\overline{y}$; $f_\alpha = \overline{X}_\alpha$, $1 \le i \le \overline{n}_1$; $f_\alpha = \overline{Z}_{\alpha-\overline{n}_1}$, $\overline{n}_1 + 1 \le \alpha \le \overline{n}_1 + \overline{n}_2$. Using (2.1) and (2.3), we compute

$$\begin{aligned}
\tau^0(h) &= g^{jj}h^0_{jj} + (1 - n_1 - 2n_2)h^0_0 y - g^{jj}h^0_j h^0_j \overline{h}^{-1}\\
&\quad + g^{jj}\sum_{\gamma=1}^{\overline{n}_1} h^\gamma_j h^\gamma_j \overline{y}^{-1} + g^{jj}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} h^\gamma_j h^\gamma_j (2\overline{y}^{-3}),\\
\tau^\alpha(h) &= g^{jj}h^\alpha_{jj} + (1 - n_1 - 2n_2)h^\alpha_0 y - 2g^{jj}h^0_j h^\alpha_j \overline{y}^{-1}\\
&\quad + g^{jj}\sum_{\beta=1}^{\overline{n}_1}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} \alpha^{\gamma-\overline{n}_1}_{\alpha\beta} h^\beta_j h^\gamma_j \overline{y}^{-2}, \qquad 1 \le \alpha \le \overline{n}_1,\\
\tau^\alpha(h) &= g^{jj}h^\alpha_{jj} + (1 - n_1 - 2n_2)h^\alpha_0 y - 4g^{jj}h^0_j h^\alpha_j \overline{y}^{-1},\\
&\qquad\qquad \overline{n}_1 + 1 \le \alpha \le \overline{n}_1 + \overline{n}_2.
\end{aligned} \tag{2.5}$$

Here j is summed from 0 to $n_1 + n_2$. Note that $\dim M = n_1 + n_2 + 1$. If the domain M is hyperbolic space, the formulas (2.5) hold with $n_2 = 0$. For $\overline{M}$

of constant negative curvature, we have the analogous formulas

$$\begin{aligned} \tau^0 &= g^{jj}h^0_{jj} + (1-n_1-2n_2)h^0_0 y - g^{jj}h^0_j h^0_j \overline{y}^{-1} \\ &\quad + g^{jj}\sum_{\gamma=1}^{\overline{n}_1} h^\gamma_j h^\gamma_j \overline{y}^{-1}, \\ \tau^\alpha(h) &= g^{jj}h^\alpha_{jj} + (1-n_1-2n_2)h^\alpha_0 y - 2g^{jj}h^0_j h^\alpha_j \overline{y}^{-1}, \qquad 1 \le \alpha \le \overline{n}_1. \end{aligned} \tag{2.5a}$$

3. Necessary conditions and uniqueness

Suppose that M is a simply connected, rank one, symmetric space of noncompact type. The exponential map, from any basepoint, provides a diffeomorphism between M and a Euclidean space with the dimension of M. One compactifies M by adding a sphere at infinity. The compactification M' of M is thus homeomorphic to a Euclidean ball of the same dimension as M. Moreover, this compactification M' admits the structure of a C^∞ manifold with boundary. The boundary coordinate charts are given by the Cayley transform. In such charts, the metric admits the representation (2.2), with the ideal boundary portion contained in $0 \times N$.

Let $h\colon M \to \overline{M}$ be a C^2 proper map between rank one symmetric spaces of noncompact type. Suppose that h extends to a C^1 map $h\colon M' \to \overline{M}'$ from the compactification M' of M, to the compactification $\overline{M}'$ of $\overline{M}$. We plan to investigate necessary conditions satisfied by the first derivatives of h at, the boundary, when h is harmonic in the interior M. We begin with some preparatory lemmas:

Lemma 3.1. *Assume that V_j are n linearly independent C^∞ vector fields defined on a ball, centered at p, in n-dimensional Euclidean space. Given real numbers α_j, there exists a C^∞ function ψ so that $V_j\psi(p) = \alpha_j$ and $V_jV_j\psi(p) = 0$, for each fixed $j = 1, 2, \ldots, n$.*

Proof. If x_k are local coordinates, then we may write $V_j = \sum_k a_{jk}(x)(\partial/\partial x_k)$, where a_{jk} is an invertible matrix. The first derivatives of ψ are determined by $\sum_k a_{jk}(p)\partial\psi(p)/\partial x_k = \alpha_j$, that is $\partial\psi(p)/\partial x_k = \sum a^{-1}_{ks}(p)\alpha_s$.

For the conditions on the second derivatives, one has

$$0 = V_jV_j\psi(p) = \sum_k a_{jk}\frac{\partial}{\partial x_k}\sum_s a_{js}\frac{\partial}{\partial x_s}\psi,$$

$$0 = \sum_{k,s} a_{jk}a_{js}\frac{\partial^2\psi}{\partial x_k\partial x_s} + \sum_k a_{jk}\frac{\partial a_{js}}{\partial x_k}\frac{\partial\psi}{\partial x_s}.$$

Define

$$\beta_j = -\sum_k a_{jk}\frac{\partial a_{js}}{\partial x_k}\frac{\partial\psi}{\partial x_s},$$

evaluated at p. Let b denote a diagonal matrix with entries $b_{jj} = \beta_j$. The condition $V_jV_jg(p) = 0$ may be written as $(a(\operatorname{Hess}\psi)a^t)_{jj} = b_{jj}$, where a^t is the transpose of a. It suffices to choose $\operatorname{Hess}\psi = a^{-1}b(a^t)^{-1}$, a symmetric matrix.

We apply the preceding lemma in a coordinate chart centered at a boundary point p of the compactification M' of M. In the unbounded model the metric is given by (2.2) and we may choose $p = (0, e) \in R \times N$, where e is the identity element in the group N. The collection of vector fields $V_j = e_j$ consists of $\partial/\partial y$, X_k, Z_l, with $1 \le k \le n_1$, $n_1 + 1 \le l \le n_1 + n_2$, and $0 \le j \le n_1 + n_2$. The Laplacian of M, acting on functions, has the form

$$\Delta\psi = \sum_j g^{jj} e_j e_j \psi + (1 - n_1 - 2n_2) y \frac{\partial \psi}{\partial y}.$$

More generally, if $\phi = \sum_i \phi_i e_i^*$ is a 1-form, then the divergence of ϕ is given by

$$\delta\phi = \sum_j g^{jj} e_j \phi_j + (1 - n_1 - 2n_2) y \phi_0.$$

If $\phi = d\psi$, then $\Delta\psi = \delta d\psi = \delta\phi$. Under the circumstances, one has

Lemma 3.2. *Suppose that $\phi \in C^1\Lambda^1 M \cap C^0\Lambda^1 M'$, is a 1-form defined on a neighborhood of $p \in M'$. If $\phi = \sum_i \phi_i e_i^*$, then there is a sequence of points $q_k \to p$, with $\sum_j g^{jj}(e_j\phi_j) y^{-1} \to 0$.*

Proof. If $\phi \in C^1\Lambda^1 M'$, the conclusion holds for any sequence converging to p, since $g^{jj} = O(y^2)$, $0 \le j \le n_1 + n_2$. Under the weaker hypothesis of the lemma, $\psi \in C^1\Lambda^1 M \cap C^0\Lambda^1 M'$, more argument is required. By Lemma 3.1, we may choose a C^∞ function ψ with $d\psi(p) = \phi(p)$ and $e_j e_j \psi(p) = 0$, for all $0 \le j \le n_1 + n_2$. let $p_k \to p$ be any sequence and use the symbol $B(p_k, 1)$ to denote the unit ball centered at p, relative to the complete metric (2.2).

By Stokes' theorem,

$$\int_{B(p_k, 1)} \delta\phi = \int_{\partial B(p_k, 1)} \phi(\nu) = \int_{\partial B(p_k, 1)} d\psi(\nu) + \varepsilon_1 y$$

where ν is a unit outward normal to $\partial B(p_k, 1)$. The symbols ε_i will denote quantities which become arbitrarily small as $p_k \to p$. The factor y appears because we measure the length of covectors in the invariant metric of the symmetric space.

Applying Stokes' theorem again,

$$\int_{B(p_k, 1)} \delta\phi = \int_{B(p_k, 1)} \Delta\psi + \varepsilon_1 y.$$

By Lemma 3.1,

$$\Delta\psi = (1 - n_1 - 2n_2)\phi\left(\frac{\partial}{\partial y}\right) + \varepsilon_2 y.$$

Consequently, by the formula for $\delta\phi$ given above,

$$\frac{1}{y}\int_{B(p_k, 1)} \sum_j g^{jj} e_j \phi_j \to 0, \qquad \text{as } p_k \to p.$$

Since the balls $B(p_k, 1)$ have volume independent of k, there exists a sequence $q_k \in B(p_k, 1)$ satisfying the conclusion of Lemma 3.2.

We return to our map $h \in C^2(M, \overline{M}) \cap C^1(M', \overline{M}')$. Formula (2.5) gives the $\partial/\partial y$ component of the tension of h. If $\tau^0(h) = O(y^{-1+\varepsilon})$, then multiply the formula, (2.5), for $\tau^0(h)$ by $\overline{y}^3 y^{-2}$, and let $y \to 0$, to deduce

Condition 3.3. $\sum_{j=0}^{n_1}\sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} h_j^\gamma h_j^\gamma = 0$, *at the boundary.*

Here, we applied Lemma 3.2 to $\phi = \sum h_j^0 e_j^*$, in order to eliminate the terms involving second derivatives. In particular, a notable special case is

Proposition 3.4. *Suppose that* $h\colon M \to \overline{M}$ *is* C^2 *proper harmonic map which extends to a* C^1 *map* $h\colon M' \to \overline{M}'$. *Let* $f\colon \partial M' \to \partial M'$ *be the boundary values of* h. *Then* $\sum_{j=1}^{n_1}\sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} f_j^\gamma f_j^\gamma = 0$.

If h is a harmonic self-map of the rank one Hermitian space, the unit ball in C^n with its Bergman metric, then Proposition 3.4 states that the boundary value of h is a contact transformation. This example is typical of the situation where the range is a rank one symmetric space, but not the hyperbolic space.

If $\overline{M}$ is hyperbolic, then Condition 3.3 is vacuous, and we now consider this situation. Suppose $h \in C^2(M, \overline{M}) \cap C^1(M', \overline{M}')$ and $\overline{M}$ has constant negative curvature. The formulas (2.5a) are now applicable. If $\tau^\alpha(h) = O(y^{1+\varepsilon})$, $\alpha \geq 0$, for some $\varepsilon > 0$, then multiply (2.5a) by $\bar{y}y^{-2}$ and let $y \to 0$ to deduce

$$(-n_1 - 2n_2)(h_0^0)^2 + \sum_{j=0}^{n_1}\sum_{\gamma=1}^{\bar{n}_1} h_j^\gamma h_j^\gamma = 0,$$
$$(-1 - n_1 - 2n_2)h_0^\alpha h_0^0 = 0, \qquad \alpha \geq 1.$$

Again, we employed Lemma 3.2 to handle the second order terms. Also, note that $h_j^0 = 0$ at the boundary, for $j \geq 1$, since $h\colon \partial M' \to \partial \overline{M}'$. It is easy to deduce

Condition 3.3a. *If the range* $\overline{M}$ *has constant negative curvature and the boundary values* f *satisfy* $\sum_{j=1}^{n_1}\sum_{\gamma=1}^{\bar{n}_1} f_j^\gamma f_j^\gamma > 0$, *then, at the boundary,*

$$h_0^0 = \left[\sum_{j=1}^{n_1}\sum_{\gamma=1}^{\bar{n}_1} \frac{f_j^\gamma f_j^\gamma}{n_1 + 2n_2}\right]^{1/2}; \qquad h_0^\alpha = 0, \ \alpha \geq 1.$$

This leads to the following uniqueness theorem.

Proposition 3.5. *Let* h *and* $\hat{h}$ *be proper harmonic maps from the rank one symmetric space* M *to the hyperbolic space* $\overline{M}$, *of constant negative curvature. Assume that both* h *and* $\hat{h}$ *extend to maps in* $C^1(M', \overline{M}')$. *If* h, $\hat{h}$ *have the same boundary map* $f\colon \partial M' \to \partial\overline{M}$, *and* $\sum_{j=1}^{n_1}\sum_{\gamma=1}^{\bar{n}_1} f_j^\gamma f_j^\gamma > 0$ *then* h *and* $\hat{h}$ *are identical.*

Proof. If both M and $\overline{M}$ are hyperbolic, this was proved by Li and Tam [8]. Their proof extends with only minor changes.

Consider again a proper map between arbitrary rank one symmetric spaces M, $\overline{M}$ of noncompact type. To proceed further, we make the additional assumption that our map h is C^2 up to the boundary of the compactification. If $\tau^0(h) = O(y^{1+\varepsilon})$, then Condition 3.3 holds, and we may multiply (2.5) by $\bar{y}^3 y^{-4}$, letting $y \to 0$ to give

Condition 3.6.

$$(n_1+2n_2)(h_0^0)^4 - \sum_{j=0}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} h_j^\gamma h_j^\gamma (h_0^0)^2 - 2\sum_{j=0}^{n_1}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} h_{j0}^\gamma h_{j0}^\gamma$$
$$-2\sum_{j=n_1+1}^{n_1+n_2}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} h_j^\gamma h_j^\gamma = 0,$$

at the boundary.

Similarly, for $1 \le \alpha \le \overline{n}_1$, we suppose that $\tau^\alpha(h) = O(y^{1+\varepsilon})$. Multiplying (2.5) by $\overline{y}^2 y^{-3}$ and letting $y \to 0$ yields, assuming our previously established Condition 3.3:

Condition 3.7.

$$(1+n_2+2n_2)h_0^\alpha (h_0^0)^2 - \sum_{j=0}^{n_1}\sum_{\beta=1}^{\overline{n}_1}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} \alpha_{\alpha\beta}^{\gamma-\overline{n}_1} h_j^\beta h_{j0}^\gamma = 0,$$

at the boundary.

Finally, we consider $\overline{n}_1+1 \le \alpha \le \overline{n}_1+\overline{n}_2$ and suppose that $\tau^\alpha(h) = O(y^{2+\varepsilon})$. Multiplying (2.5) by $\overline{y}y^{-3}$, using Condition 3.3, and letting $y \to 0$, gives

Condition 3.8. $(2+n_1+2n_2)h_0^0 h_{00}^\alpha = 0$, *at the boundary.*

Note that $h_j^\alpha = 0$, along the boundary, by condition 3.3, for $1 \le j \le n_1$, $\overline{n}_1 + 1 \le \alpha \le \overline{n}_1 + \overline{n}_2$. Since e_j is tangent to the boundary, we also have $h_{jj}^\alpha = e_j h_j^\alpha = 0$, for such j and α.

To proceed further, we consider the integrability condition $ddh = 0$, which holds for any C^2 map h [2]. Recall that $dh = h_i^\alpha e_i^* \otimes f_\alpha$, and

$$\nabla dh = h_{ij}^\alpha e_j^* \otimes e_i^* \otimes f_\alpha + h_i^\alpha e_j^* \otimes \nabla_{e_j} e_i^* \otimes f_\alpha + h_i^\alpha e_j^* \otimes e_i^* \otimes \nabla_{e_j} f_\alpha .$$

Thus

$$0 = ddh = h_{ij}^\alpha e_j^* \wedge e_i^* \otimes f_\alpha + h_i^\alpha e_j^* \wedge \nabla_{e_j} e_i^* \otimes f_\alpha + h_i^\alpha e_j^* \wedge e_i^* \otimes \nabla_{e_j} f_\alpha .$$

However, $\nabla_{e_j} e_i^* = e_k(\nabla_{e_j} e_i^*) e_k^* = -e_i^*(\nabla_{e_j} e_k) e_k^*$ and thus

$$h_{ij}^\alpha e_j^* \wedge e_i^* \otimes f_\alpha - h_i^\alpha e_i^*(\nabla_{e_j} e_k) e_j^* \wedge e_k^* \otimes f_\alpha + h_i^\gamma h_j^\beta e_j^* \wedge e_i^* \otimes \nabla_{f_\beta} f_\gamma = 0 .$$

This may be rewritten as

$$h_{ij}^\alpha e_j^* \wedge e_i^* \otimes f_\alpha - \tfrac{1}{2} h_i^\alpha e_i^*([e_j, e_k]) e_j^* \wedge e_k^* \otimes f_\alpha + \tfrac{1}{2} h_i^\gamma h_j^\beta e_j^* \wedge e_i^* \otimes [f_\beta, f_\gamma] = 0$$

or equivalently

$$h_{ij}^\alpha e_i^* \wedge e_j^* = -\tfrac{1}{2} h_i^\alpha e_i^*([e_j, e_k]) e_j^* \wedge e_k^* + \tfrac{1}{2} h_i^\beta h_j^\gamma f_\alpha^*([f_\beta, f_\gamma]) e_i^* \wedge e_j^* . \tag{3.9}$$

Note that the expression h_{k0}^α for $1 \le k \le n_1$, $\overline{n}_1 + 1 \le \alpha \le \overline{n}_1 + \overline{n}_2$, occurs in the Condition 3.7. By Condition 3.3, one has $h_{0k}^\alpha = 0$, since e_k is tangent to the boundary. So, we may deduce from (3.9) that

$$h_{k0}^\alpha = \tfrac{1}{2}(h_k^\beta h_0^\gamma - h_0^\beta h_k^\gamma) f_\alpha^*([f_\beta, f_\gamma]) = \sum_{\beta=1}^{\overline{n}_1}\sum_{\gamma=1}^{\overline{n}_1} h_k^\beta h_0^\gamma a_{\beta\gamma}^{\alpha-\overline{n}_1} . \tag{3.10}$$

Relabeling the indices and substituting back into Condition 3.7 gives $1 \le \alpha \le \overline{n}_1$,

Condition 3.11.

$$(1+n_1+2n_2)h_0^\alpha(h_0^0)^2+\sum_{j=0}^{\overline{n}_1}\sum_{\beta=1}^{\overline{n}_1}\sum_{\varepsilon=1}^{\overline{n}_1}\sum_{\mu=1}^{\overline{n}_1}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2}\alpha_{\beta\alpha}^{\gamma-\overline{n}_1}h_j^\beta h_j^\varepsilon\alpha_{\varepsilon\mu}^{\gamma-\overline{n}_1}h_0^\mu=0$$

at the boundary.

In preparation for our uniqueness theorem, we deduce

Lemma 3.12. *Let* $h\in C^2(M',\overline{M}')$ *satisfy* 3.3, 3.6, 3.7, 3.8 *on* M. *Assume that the boundary data* f *satisfies* $\sum_{j=n_1+1}^{n_1+n_2}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2}f_j^\gamma f_j^\gamma>0$ *then* $h_0^0>0$, $h_0^\alpha=0$ *for* $1\le\alpha\le\overline{n}_1+\overline{n}_2$, *and* $h_{k0}^\beta=h_{00}^\beta=0$ *for* $\overline{n}_1+1\le\beta\le\overline{n}_1+\overline{n}_2$, $1\le k\le n_1$.

Proof. The inequality $h_0^0>0$ follows immediately from Condition 3.6 and our hypothesis about the boundary data. If $\overline{n}_1+1\le\alpha\le\overline{n}_1+\overline{n}_2$, then $h_0^\alpha=0$, by Condition 3.3.

If $1\le\alpha\le\overline{n}_1$, then we multiply Condition 3.11 by h_0^α and sum over α, yielding

$$(1+n_1+2n_2)(h_0^0)^2\sum_{\alpha=1}^{\overline{n}_1}(h_0^\alpha)^2+\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2}\sum_{j=0}^{\overline{n}_1}\left(\sum_{\varepsilon,\mu=1}^{\overline{n}_1}h_j^\varepsilon a_{\varepsilon\mu}^{\gamma-\overline{n}_1}h_0^\mu\right)^2=0.$$

Since all terms in the sum are positive, $h_0^\alpha=0$.

It now follows from (3.10) that $h_{k0}^\beta=0$, for $1\le k\le n_1$ and $\overline{n}_1+1\le\beta\le\overline{n}_1+\overline{n}_2$, since we have just shown that $h_0^\gamma=0$, for $1\le\gamma\le\overline{n}_1$. Condition 3.8 implies $h_{00}^\beta=0$, when $\overline{n}_1+1\le\beta\le\overline{n}_1+\overline{n}_2$.

This allows us to derive the following uniqueness result.

Theorem 3.13. *Let* h *and* $\hat{h}$ *be proper harmonic maps between rank one symmetric spaces of noncompact type. Assume that both* h *and* $\hat{h}$ *extend to maps in* $C^2(M',\overline{M}')$. *If* h, $\hat{h}$ *both have the same boundary map* $f\colon\partial M'\to\partial\overline{M}'$ *and* $\sum_{j=n_1+1}^{n_1+n_2}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2}f_j^\gamma f_j^\gamma>0$ *then* $h=\hat{h}$, *everywhere.*

Proof. Let $(y,n)\in R^+\times N$ and $(\overline{y},\overline{n})\in R^+\times\overline{N}$ be local coordinates near the boundary of M' and $\overline{M}'$. If d denotes the Riemannian distance in $\overline{M}$, then

$$\begin{aligned}d(h,\hat{h})\le{}& d(h,(\overline{y}(h),\overline{n}(f)))+d((\overline{y}(h),\overline{n}(f)),(\overline{y}(\hat{h}),\overline{n}(f)))\\&+d((\overline{y}(\hat{h}),\overline{n}(f)),\hat{h}).\end{aligned}$$

To estimate the first term, we consider the curve $(\overline{y}(h),\overline{n}(h(t,n)))$, $0\le t\le y$, which joins $(\overline{y}(h),\overline{n}(f))$ to $h=h(y,n)$. One has

$$d(h,(\overline{y}(h),\overline{n}(f)))\le c_1\int_0^y\left(\overline{y}^{-1}\sum_{\alpha=1}^{\overline{n}_1}|h_0^\alpha|+\overline{y}^{-2}\sum_{\alpha=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2}|h_0^\alpha|\right)dt.$$

By Lemma 3.12, $|h_0^\alpha|=O(t)$, for $1\le\alpha\le\overline{n}_1+\overline{n}_2$; $|h_0^\alpha|=o(t)$, for $\overline{n}_1+1\le\alpha\le\overline{n}_1+\overline{n}_2$. Since $h_0^0>0$, y is comparable to $\overline{y}$. So

$$d(h,(\overline{y}(h),\overline{n}(f)))\le c_2\left(y^{-1}\int_0^y t\,dt+y^{-2}\int_0^y o(t)\,dt\right).$$

So $d(h, (\overline{y}(h), \overline{n}(f))) = o(1)$. The third term is completely analogous.

To estimate the second term, we employ the curve $(t, \overline{n}(f))$, $\overline{y}(h) \le t \le \overline{y}(\hat{h})$. Condition 3.6 implies that $h_0^0 = \hat{h}_0^0$. This is because Lemma 3.12 implies $h_{j0}^\gamma = \hat{h}_{j0}^\gamma = 0$, $0 \le j \le n_1$, $\overline{n}_1 + 1 \le \gamma \le \overline{n}_1 + \overline{n}_2$, and for $1 \le \alpha \le \overline{n}_1$, $h_0^\alpha = \hat{h}_0^\alpha = 0$. So

$$d((\overline{y}(h), \overline{n}(f)), (\overline{y}(\hat{h}), \overline{n}(f))) = \left|\int_{\overline{y}(h)}^{\overline{y}(\hat{h})} \frac{dt}{t}\right| = \left|\ln\left(\frac{\overline{y}(\hat{h})}{\overline{y}(h)}\right)\right|$$
$$= \left|\ln\left(\frac{h_0^0 y + o(y)}{h_0^0 y + o(y)}\right)\right| = o(1).$$

Thus $d^2(h, \hat{h})$ is a subharmonic function, which vanishes on $\partial M'$. it must be identically zero. The subharmonicity is well known, since the range has nonpositive curvature [10].

4. Existence

The necessary conditions of the previous section lead naturally to a construction of harmonic maps, given sufficiently regular boundary data. Assume that M and $\overline{M}$ are two rank one symmetric spaces of noncompact type, with their compactifications M' and $\overline{M}'$. Let $f: \partial M' \to \partial \overline{M}'$ be a $C^{2,\varepsilon}$ map, $0 < \varepsilon < 1$, satisfying $f_j^\gamma = 0$, $1 \le j \le n_1$, $\overline{n}_1 + 1 \le \gamma \le \overline{n}_1 + \overline{n}_2$. Our goal is to construct a harmonic map $M \to \overline{M}$, which assumes the boundary values f. This will be achieved by constructing a map h, with boundary data f, whose tension field decays to zero at infinity, and then applying the nonlinear heat equation to deform our approximate solution to a harmonic map.

Our first step is to establish a converse to the necessary conditions of §3.

Lemma 4.1. *Let $h \in C^{2,\varepsilon}(M', \overline{M}')$, $0 < \varepsilon < 1$. Such h satisfies the following conditions, at the boundary*

(i)
$$\sum_{j=0}^{n_1} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} h_j^\gamma h_j^\gamma = 0,$$

(ii)
$$(n_1 + 2n_2)(h_0^0)^4 - \sum_{j=0}^{n_1} \sum_{\gamma=1}^{\overline{n}_1} h_j^\gamma h_j^\gamma (h_0^0)^2$$
$$- 2\sum_{j=0}^{n_1} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} h_{j0}^\gamma h_{j0}^\gamma - 2 \sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} h_j^\gamma h_j^\gamma = 0,$$

(iii)
$$(1 + n_1 + 2n_2) h_0^\alpha (h_0^0)^2 - \sum_{j=0}^{n_1} \sum_{\beta=1}^{\overline{n}_1} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} a_{\alpha\beta}^{\gamma-\overline{n}_1} h_j^\beta h_{j0}^\gamma = 0, \qquad 1 \le \alpha \le \overline{n}_1,$$

(iv)
$$h_0^0 h_{00}^\alpha = 0, \qquad \overline{n}_1 + 1 \le \alpha \le \overline{n}_1 + \overline{n}_2,$$

whenever $\tau^0(h) = O(y^{1+\varepsilon})$; $\tau^\alpha(h) = O(y^{1+\varepsilon})$, $1 \le \alpha \le \bar{n}_1$; $\tau^\alpha(h) = O(y^{2+\varepsilon})$, $\bar{n}_1+1 \le \alpha \le \bar{n}_1+\bar{n}_2$. *Conversely, if conditions* (i)–(iv) *hold, then the components* $\tau^\alpha(h)$, *of the tension field, have the indicated decay as* $y \downarrow 0$, *provided* $h_0^0 > 0$.

Proof. In §3, we established (i)–(iv), for any $h \in C^2(M', \overline{M}')$, whose tension field decays as supposed. The converse assertion follows from (2.5). If $h_0^0 > 0$, then the second order Taylor expansion, of h, gives corresponding approximations for the components of $\tau(h)$. Conditions (i) and (ii) force the vanishing of the first two terms approximating $\tau^0(h)$, the remainder is of order $y^{1+\varepsilon}$. Conditions (iii), (i) imply that the lead two terms for $\tau^\alpha(h)$, $1 \le \alpha \le \bar{n}_1$, are zero, so $\tau^\alpha(h) = O(y^{1+\varepsilon})$. Lastly, conditions (iv), (i) imply that the first two terms for $\tau^\alpha(h)$, $\bar{n}_1 + 1 \le \alpha \le \bar{n}_1 + \bar{n}_2$, are zero, so $\tau^\alpha(h) = O(y^{2+\varepsilon})$.

Next, we construct an asymptotically harmonic map, with appropriately given boundary values:

Proposition 4.2. *Suppose that* $f \in C^{2,\varepsilon}(\partial M', \partial\overline{M}')$, $0 < \varepsilon < 1$, *satisfies* $f_j^\gamma = 0$, $1 \le j \le n_1$, $\bar{n}_1 + 1 \le \gamma \le \bar{n}_1 + \bar{n}_2$, *and* $\sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} f_j^\gamma f_j^\gamma > 0$. *Then there exists* $h \in C^{2,\varepsilon}(M', \overline{M}')$, *assuming the boundary values* f *continuously, with* $\|\tau(h)\| = O(y^\varepsilon)$. *Here* $\|\tau(h)\|$ *is the norm of the tension field in the Riemannian norm.*

Proof. Motivated by (ii) of Lemma 4.1, we let $\phi > 0$ be a solution of

$$(n_1 + 2n_2)\phi^4 - \sum_{j=1}^{n_1} \sum_{\gamma=1}^{\bar{n}_1} f_j^\gamma f_j^\gamma \phi^2 - 2 \sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} f_j^\gamma f_j^\gamma = 0.$$

In our local chart near the boundary, we extend ϕ, by convolving with a smoothing kernel, commensurable to the Euclidean Poisson kernel. Since $f \in C^{2,\varepsilon}$, ϕ and its extension lies in $C^{1,\varepsilon}$, moreover [11], $|\nabla_0^2\phi| = O(y^{\varepsilon-1})$, by an elementary Poisson kernel estimate. Here $|\nabla_0^2\phi|$ is a locally defined Euclidean norm, in our chart. Define $h(y, n) = (y\phi(y, n), f(n))$. Then $h \in C^{2,\varepsilon}$.

For this h, $h_0^0 = \phi$; $h_0^\alpha = 0$, $1 \le \alpha \le \bar{n}_1+\bar{n}_2$, $h_{00}^\alpha = 0$, $\bar{n}_1+1 \le \alpha \le \bar{n}_1+\bar{n}_2$; $h_{j0}^\gamma = 0$, $1 \le j \le n_1$, $\bar{n}_1 + 1 \le \gamma \le \bar{n}_1 + \bar{n}_2$, and h restricts to f on $\partial M'$. Thus, conditions (i)–(iv) of Lemma 4.1 hold at the boundary. By Lemma 4.1 and the expression (2.2) of the metric $\|\tau(h)\| = O(y^\varepsilon)$. This completes the local construction. Since $\|\tau(h)\| = O(y^\varepsilon)$, and Lemma 4.1 is an equivalence statement, (i)–(iv) are valid in any coordinate patch. Thus, the conclusion of Lemma 3.12 holds, not just in the chart where h was constructed, but in any overlapping chart. We now fit together our local solutions via partition of unity, along the boundary. The partition functions can be chosen independent of y, near $\partial\overline{M}'$. The conclusion of Lemma 3.12 is seen to hold for our global solution. However, this implies (i)–(iv) of Lemma 4.1 and thus $\|\tau(h)\| = O(y^\varepsilon)$.

The deformation, via the nonlinear heat equation, employs certain superharmonic functions as barriers. In standard notation, let r denote the geodesic distance from the basepoint in our rank one symmetric space M. One has

Lemma 4.3. *Assume that* r_0 *is sufficiently large. Define, for any given* $0 < s \le n_1 + 2n_2$, $\psi(r) = e^{-sr}$, $r \ge r_0$, *and* $\psi(r) = e^{-sr_0}$, $r \le r_0$. *Then* ψ *is superharmonic, on* M.

Proof. In exponential polar coordinates (r, w), the volume element is written as $(\sinh r)^{n_1}(\sinh 2r)^{n_2}\,dr\,dw$. To check our normalization of metric, observe that r is commensurable to $-\ln y$ in (2.2). If $r \geq r_0$, the standard expression for the Laplacian of a radial function gives

$$\begin{aligned}\Delta\psi(r) &= \psi''(r) + \left(n_1\frac{\cosh r}{\sinh r} + 2n_2\frac{\cosh 2r}{\sinh 2r}\right)\psi'(r)\\ &= s\left(s - n_1\frac{\cosh r}{\sinh r} - 2n_2\frac{\cosh 2r}{\sinh 2r}\right)\psi(r).\end{aligned}$$

So $\Delta\psi(r) \leq 0$, because $0 < s \leq n_1 + 2n_2$. Since the minimum of two superharmonic functions is superharmonic and superharmonic is a local concept, ψ is superharmonic on all of M.

Combining Proposition 4.2, Lemma 4.3, and the method of Li and Tam [8], one deduces

Theorem 4.4. *Suppose that* $f \in C^{2,\varepsilon}(\partial M', \partial\overline{M}')$, $0 < \varepsilon < 1$, *satisfies* $f_j^\gamma = 0$, $1 \leq j \leq n_1$, $\overline{n}_1 + 1 \leq \gamma \leq \overline{n}_1 + \overline{n}_2$, *and* $\sum_{j=n_1+1}^{n_1+n_2}\sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} f_j^\gamma f_j^\gamma > 0$. *Then there exists a harmonic map* $u\colon M \to \overline{M}$, *which assumes the boundary values* f, *continuously. If* h *is the map of Proposition* 4.2, *then the Riemannian distance from* h *to* u *is* $O(y^{\overline{\varepsilon}})$, *in any standard local chart near the boundary, for any* $\overline{\varepsilon} < \varepsilon$.

Proof. Suppose that u_t is the solution to the nonlinear heat equation with initial data h. Since $\|\tau(h)\|^2$ lies in some L^p, $p > 1$, $\tau(h)$ is bounded, and h has bounded energy density, it follows [7] that u_t exists and converges to a harmonic map $u = u_\infty$, as $t \to \infty$. Hartman [5] showed that $\|u_t\|$ is a subsolution to the usual linear heat equation. Choosing $s = \varepsilon$, in Lemma 4.3, we get an infinite number of subsolutions $\|u_t\| - c\psi$, any $c \geq 0$. If c is large enough, then, at $t = 0$, $\|u_t\| - c\psi = \|\tau(h)\| - c\psi < 0$, by the decay estimate for $\tau(h)$ in Proposition 4.2. The maximum principle gives $\|\tau(u_t)\| < c\psi$, for all t. The general existence theorem of [7] states that $\|\tau(u_t)\| < c_1e^{-c_2t}$, for some positive constants c_1 and c_2.

Thus, for any T,

$$d(h, u) \leq \int_0^\infty \|u_t\|\,dt = \int_0^T \|u_t\|\,dt + \int_0^\infty \|u_t\|\,dt.$$

The conclusion follows by choosing T of order $-\log\psi$, for points near the ideal boundary at infinity, $\partial M'$.

Suppose that the image $\overline{M}$ is a hyperbolic space of constant negative curvature -1. In this case, the regularity requirement of Theorem 4.4 may be significantly lowered. The analogue of Lemma 4.1 is

Lemma 4.5. *Suppose* $h \in C^{1,\varepsilon}(M', \overline{M}') \cap C^2(M, \overline{M})$, $0 < \varepsilon < 1$, *with* $\overline{M}$ *of constant negative curvature* -1. *Such* h *satisfies the following conditions, at the boundary*:

(i) $(n_1 + 2n_2)(h_0^0)^2 - \sum_{j=0}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} h_j^\gamma h_j^\gamma = 0$,
(ii) $h_0^0 h_0^\alpha = 0$, $\alpha \geq 1$,

whenever $\tau^\alpha(h) = O(y^{1+\varepsilon})$, $\alpha \geq 0$. *Conversely, suppose that for the Euclidean norm, in any local coordinate chart,* $|\nabla^2 h| = O(y^{\varepsilon-1})$. *If* $h_0^0 > 0$ *and conditions* (i), (ii) *both hold, then* $\tau^\alpha(h) = O(y^{1+\varepsilon})$, $\alpha \geq 0$.

Proof. If $\tau^\alpha(h) = O(y^{1+\varepsilon})$, then we proved (i), (ii) in §3, for $h \in C^1(M', \overline{M}') \cap C^2(M, \overline{M})$. Conversely, the hypothesis $|\nabla^2 h| = O(y^{\varepsilon-1})$, shows that the second derivative terms in (2.5a) are of order $y^{1+\varepsilon}$. If $h_0^0 > 0$, then (i) gives the vanishing of the first derivative terms, in formula (2.5a) for $\tau^0(h)$, up to order $y^{1+\varepsilon}$. Similarly, (ii) handles the first derivative terms for $\tau^\alpha(h)$, $\alpha \geq 1$.

Following our earlier scheme, we construct an asymptotically harmonic map, given appropriate boundary data.

Proposition 4.6. *Assume that* $f \in C^{1,\varepsilon}(\partial M', \partial \overline{M})$, $0 < \varepsilon < 1$, *satisfies*

$$\sum_{j=1}^{n_1} \sum_{\gamma=1}^{\overline{n}_1} f_j^\gamma f_j^\gamma > 0.$$

Then there exists $h \in C^{1,\varepsilon}(M', \overline{M}') \cap C^2(M, \overline{M})$, *assuming the boundary values* f *continuously, with* $\|\tau(h)\| = O(y^\varepsilon)$, *the Riemannian norm, of our space* $\overline{M}$ *with constant negative curvature.*

Proof. Denote

$$\phi = \left[\sum_{j=1}^{n_1} \sum_{\gamma=1}^{\overline{n}_1} \frac{f_j^\gamma f_j^\gamma}{n_1 + 2n_2} \right]^{1/2},$$

as suggested by the hypothesis (i) of Lemma 4.5. Clearly, $\phi \in C^{0,\varepsilon}(\partial M')$, and we extend ϕ locally by convolving with the smoothing kernel, comparable to the Poisson kernel. In contrast to the proof of Proposition 4.2, we only have $f \in C^{1,\varepsilon}$. So we must also extend f, by convolution with a kernel comparable to the Poisson kernel, using the components of f in some chart near $\partial \overline{M}'$. We now define

$$h(y, n) = \left(y\phi(y, n), f(y, n) - \frac{\partial f}{\partial y}(0, n) y \right).$$

Elementary estimates for Poisson smoothing [11], now show that $h \in C^{1,\varepsilon}(M', \overline{M}')$, as in the proof of Proposition 4.2. Moreover, $h_0^0 = \phi$ and $h_0^\alpha = 0$, $\alpha \geq 1$, at the boundary $\partial M'$. Since h has boundary values f, $h(0, n) = (0, f(0, n))$, conditions (i), (ii) of Lemma 4.5 are valid. The Poisson smoothing guarantees that $|\nabla^2 h| = O(y^{\varepsilon-1})$. Thus, Lemma 4.5 yields $\|\tau(h)\| = O(y^\varepsilon)$. Since $\overline{M}$ has constant negative curvature, the norm is y^{-1} times the locally defined Euclidean norm. This completes the local construction, on a chart near $\partial M'$. One patches these local solutions together using a partition of unity along $\partial M'$. The partition functions can be chosen independent of y, near $\partial M'$, so that $h_0^0 = \phi$ and $h_0^\alpha = 0$, $\alpha \geq 1$, for the globally defined h, in any local chart. Lemma 4.5 again gives $\|\tau^\alpha(h)\| = O(y^\varepsilon)$.

We now invoke Lemma 4.3 and apply the same argument as in the proof of Theorem 4.4, to deduce

Theorem 4.7. *Assume that $\overline{M}$ is the simply connected complete space having constant negative curvature -1. Let M be a rank one symmetric space of noncompact type.*

Suppose that $f \in C^{1,\varepsilon}(\partial M', \partial\overline{M}')$, $0 < \varepsilon < 1$, satisfies $\sum_{j=1}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} f_j^\gamma f_j^\gamma > 0$. Then there exists a harmonic map $u\colon M \to \overline{M}$, which assumes the boundary values f, continuously. If h is the map of Proposition 4.6, then the Riemannian distance from h to u is $O(y^{\overline{\varepsilon}})$, for any $\overline{\varepsilon} < \varepsilon$.

Remark. By applying the arguments of [9], it suffices to assume $f \in C^2(\partial M', \partial\overline{M}')$ in the hypothesis of Theorem 4.4. Similarly, one may suppose $f \in C^1(\partial M', \partial\overline{M}')$ in Theorem 4.7. We omit the details since these refinements are not needed in the subsequent sections of this paper. A more careful discussion will be given elsewhere.

5. Higher order approximate solutions and compatibility conditions

Let M and $\overline{M}$ be rank one Riemannian symmetric spaces of noncompact type. Suppose that $f \in C^{2,\varepsilon}(\partial M', \partial\overline{M}')$ satisfies the hypothesis of Proposition 4.2. We showed that there exists $h \in C^{2,\varepsilon}(M', \overline{M}')$, assuming the boundary values f continuously, whose tension field satisfies $\|\tau(h)\| = O(y^\varepsilon)$. If the boundary data is smoother, $f \in C^{l+2,\varepsilon}(\partial M', \partial\overline{M}')$, then we will modify h to achieve $\|\tau(h)\| = O(y^{l+\varepsilon})$. It is already clear, from the proof of Proposition 4.2, that the h constructed there lies in $C^{l+2,\varepsilon}(M', \overline{M}')$, whenever $f \in C^{l+2,\varepsilon}(\partial M', \partial\overline{M}')$. The point is to improve the decay rate of the tension field. One proceeds by an inductive argument, which is valid as long as $l \le n_1 + 2n_2$. The breakdown after a finite number of steps is expected by analogy with the studies of related problems in [4] and [8]. These higher order approximate solutions, besides being of intrinsic interest, play an important role in our subsequent development of regularity theory.

To set up the induction, assume that $h \in C^{l+2,\varepsilon}$, $l \ge 1$, $0 < \varepsilon < 1$, has boundary values $f \in C^{l+2,\varepsilon}$. Suppose that $h_0^0 > 0$; $h_0^\alpha = 0$, $1 \le \alpha \le \overline{n}_1 + \overline{n}_2$, $h_{00}^\alpha = 0$, $\overline{n}_1 + 1 \le \alpha \le \overline{n}_1 + \overline{n}_2$; $h_{j0}^\gamma = 0$, $1 \le j \le n_1$, $\overline{n}_1 + 1 \le \gamma \le \overline{n}_1 + \overline{n}_2$. If $k < l+2$, assume that $\partial^{i-1}h_0^\alpha/\partial y^{i-1}$ are determined for $i < k$ and all α. Moreover, suppose that $\partial^{i-1}h_0^\alpha/\partial y^{i-1}$ are determined for $i \le k$, $\alpha \ge \overline{n}_1 + 1$. These modifications have been made to achieve $|\tau^\alpha(h)| = O(y^k)$, all α; and $|\tau^\alpha(h)| = O(y^{k+1})$, $\alpha \ge \overline{n}_1 + 1$. To start the induction, with $k = 2$, we use the proofs of Propositions 4.2, 4.6. Let Q_k denote a rational function of the already determine data. Note that if $\partial^{i-1}h_0^\alpha/\partial y^{i-1}$ is determined, then so are its tangential derivatives, as long as the total number of derivatives is at most $l+2$.

The inductive argument requires some detailed calculations, starting from the formulas (2.5). It is convenient to divide the presentation into three cases, depending upon the index α in $\tau^\alpha(h)$.

Case 1. $\alpha \ge \overline{n}_1 + 1$. By formula (2.5),

$$\tau^\alpha(h) = g^{jj}h_{jj}^\alpha + (1 - n_1 - 2n_2)h_0^\alpha y - 4g^{jj}h_j^0 h_j^\alpha \overline{y}^{-1}$$

where $0 \le j \le n_1 + n_2$ is summed. Thus, by the decomposition (2.2) of the metric,

$$\tau^\alpha(h) = y^2 h^\alpha_{00} + \sum_{j=1}^{n_1} y^2 h^\alpha_{jj} + \sum_{j=n_1+1}^{n_1+n_2} y^4 h^\alpha_{jj} + (1-n_1-2n_2)h^\alpha_0 y$$
$$- 4\left(\frac{y^2}{\overline{y}}\right) h^0_0 h^\alpha_0 - 4\left(\frac{y^2}{\overline{y}}\right)\sum_{j=1}^{n_1} h^0_j h^\alpha_j - 4\left(\frac{y^4}{\overline{y}}\right)\sum_{j=n_1+1}^{n_1+n_2} h^0_j h^\alpha_j .$$

We now separate out those terms Q_k already fixed at an earlier step of the induction

$$\tau^\alpha(h) = y^2 h^\alpha_{00} + (1-n_1-2n_2)h^\alpha_0 y - 4(y^2/\overline{y})h^0_0 h^\alpha_0 + Q_k y^{k+1} + O(y^{k+1+\varepsilon}).$$

Here, we used the fact that $h^0_j = 0$, $j \geq 1$; $h^\alpha_0 = 0$, $\alpha \geq 1$, along the boundary. Using Taylor expansion of the remaining terms, we deduce, since $h^\alpha_{00} = 0$ at the boundary,

$$k!\tau^\alpha(h) = (k-n_1-2n_2-3)\frac{\partial^k h^\alpha_0}{\partial y^k} y^{k+1} + Q_k y^{k+1} + O(y^{k+1+\varepsilon}).$$

If $k+1 < l+2$, the remainder term is $O(y^{k+2})$. Since $l \leq n_1 + 2n_2$ and $k < l+2$, we may solve uniquely for $\partial^k h^\alpha_0/\partial y^k$, in terms of Q_k, to assure that $\tau^\alpha(h) = O(y^{k+1+\varepsilon})$ and $\tau^\alpha(h) = O(y^{k+2})$, as long as $k+1 < l+2$.

Case 2. $1 \leq \alpha \leq \overline{n}_1$. Again, starting from (2.5),

$$\tau^\alpha(h) = g^{jj} h^\alpha_{jj} + (1-n_1-2n_2)h^\alpha_0 y - 2g^{jj} h^0_j h^\alpha_j \overline{y}^{-1}$$
$$+ g^{jj} \sum_{\beta=1}^{\overline{n}_1} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} a^{\gamma-\overline{n}_1}_{\alpha\beta} h^\beta_j h^\gamma_j \overline{y}^{-2}$$

with j summed from 0 to $n_1 + n_2$. Using the local expression (2.2) for the metric,

$$\tau^\alpha(h) = y^2 h^\alpha_{00} + y^2 \sum_{j=1}^{n_1} h^\alpha_{jj} + y^4 \sum_{j=n_1+1}^{n_1+n_2} h^\alpha_{jj} + (1-n_1-2n_2)h^\alpha_0 y$$
$$- 2y^2 h^0_0 h^\alpha_0 \overline{y}^{-1} - 2y^2\overline{y}^{-1} \sum_{j=1}^{n_1} h^0_j h^\alpha_j - 2y^4\overline{y}^{-1} \sum_{j=n_1+1}^{n_1+n_2} h^0_j h^\alpha_j$$
$$+ y^2\overline{y}^{-2} \sum_{\beta=1}^{\overline{n}_1} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} a^{\gamma-\overline{n}_1}_{\alpha\beta} h^\beta_0 h^\gamma_0$$
$$+ y^2\overline{y}^{-2} \sum_{j=1}^{n_1} \sum_{\beta=1}^{\overline{n}_1} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} a^{\gamma-\overline{n}_1}_{\alpha\beta} h^\beta_j h^\gamma_j$$
$$+ y^4\overline{y}^{-2} \sum_{j=n_1+1}^{j=n_1+n_2} \sum_{\beta=1}^{\overline{n}_1} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} a^{\gamma-\overline{n}_1}_{\alpha\beta} h^\beta_j h^\gamma_j .$$

Identifying certain terms Q_k already fixed at an earlier inductive step:

$$\tau^\alpha(h) = y^2 h^\alpha_{00} + (1-n_1-2n_2)h^\alpha_0 y - 2(y^2/\overline{y})h^0_0 h^\alpha_0 + Q_k y^k + O(y^{k+\varepsilon}).$$

The hypotheses $h_0^\alpha = 0$, $\alpha \geq 1$; $h_{00}^\gamma = 0$, $\gamma \geq \bar{n}_1 + 1$; $h_{j0}^\gamma = 0$, $\gamma \geq \bar{n}_1 + 1$, $1 \leq j \leq n_1$, were needed here, at the boundary. Taylor expanding the relevant terms gives

$$(k-1)!\tau^\alpha(h) = (k - n_1 - 2n_2 - 2)\frac{\partial^{k-1} h_0^\alpha}{\partial y^{k-1}} y^k + Q_k y^k + O(y^{k+\varepsilon}).$$

If $k + 1 < l + 2$, then the remainder is of order $O(y^{k+1})$. Since $k < l + 2$ and $l \leq n_1 + 2n_2$, the derivative $\partial^{k-1} h_0^\alpha / \partial y^{k-1}$ is uniquely determined, in terms of the previously known data Q_k, to give $\tau^\alpha(h) = O(y^{k+\varepsilon})$ and in fact the better condition $\tau^\alpha(h) = O(y^{k+1})$, as long as $k + 1 < l + 2$.

Case 3. $\alpha = 0$. Returning to (2.5), we have

$$\begin{aligned}\tau^0(h) &= g^{jj} h_{jj}^0 + (1 - n_1 - 2n_2) h_0^0 y - g^{jj} h_j^0 h_j^0 \bar{y}^{-1} \\ &\quad + g^{jj} \sum_{\gamma=1}^{\bar{n}_1} h_j^\gamma h_j^\gamma \bar{y}^{-1} + g^{jj} \sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} h_j^\gamma h_j^\gamma (2\bar{y}^{-3})\end{aligned}$$

where one sums j from 0 to $n_1 + n_2$. Breaking this into pieces corresponding to the splitting (2.2) of the metric, we have

$$\begin{aligned}\tau^0(y) &= y^2 h_{00}^0 + \sum_{j=1}^{n_1} y^2 h_{jj}^0 + \sum_{j=n_1+1}^{n_1+n_2} y^4 h_{jj}^0 + (1 - n_1 - 2n_2) h_0^0 y \\ &\quad - y^2 (h_0^0)^2 \bar{y}^{-1} - y^2 \sum_{j=1}^{n_1} h_j^0 h_j^0 \bar{y}^{-1} - y^4 \sum_{j=n_1+1}^{n_1+n_2} h_j^0 h_j^0 \bar{y}^{-1} \\ &\quad + y^2 \sum_{\gamma=1}^{\bar{n}_1} (h_0^\gamma)^2 \bar{y}^{-1} + y^2 \sum_{j=1}^{n_1} \sum_{\gamma=1}^{\bar{n}_1} h_j^\gamma h_j^\gamma \bar{y}^{-1} \\ &\quad + y^4 \sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=1}^{\bar{n}_1} h_j^\gamma h_j^\gamma \bar{y}^{-1} + 2 \sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} h_0^\gamma h_0^\gamma y^2 \bar{y}^{-3} \\ &\quad + 2y^2 \sum_{j=1}^{n_1} \sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} h_j^\gamma h_j^\gamma \bar{y}^{-3} + 2y^4 \sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} h_j^\gamma h_j^\gamma \bar{y}^{-3}.\end{aligned}$$

Isolating appropriate terms Q_k which are previously determined,

$$\begin{aligned}\tau^0(h) &= y^2 h_{00}^0 + (1 - n_1 - 2n_2) h_0^0 y - y^2 (h_0^0)^2 \bar{y}^{-1} \\ &\quad + y^2 \bar{y}^{-1} \sum_{j=1}^{n_1} \sum_{\gamma=1}^{\bar{n}_1} h_j^\gamma h_j^\gamma + 2y^4 \bar{y}^{-3} \sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} h_j^\gamma h_j^\gamma + Q_k y^k + O(y^{k+\varepsilon}).\end{aligned}$$

Again, we used the facts $h_0^\alpha = 0$, $\alpha \geq 1$; $h_{00}^\gamma = 0$, $h_{j0}^\gamma = 0$, $\gamma \geq \bar{n}_1 + 1$, $1 \leq j \leq n_1$.

Using Taylor polynomials to estimate each remaining term gives

$$k!\tau^0(h) = \left[1 + k(k - n_1 - 2n_2 - 2) - \left(\frac{1}{h_0^0}\right)^2 \sum_{j=1}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} h_j^\gamma h_j^\gamma - \frac{6}{(h_0^0)^4} \sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} h_j^\gamma h_j^\gamma \right] \frac{\partial^{k-1} h_0^0}{\partial y^{k-1}} y^k + Q_k y^k + O(y^{k+\varepsilon}).$$

If $k+1 < l+2$, the remainder is $O(y^{k+1})$. Since $l \le n_1 + 2n_2$ and $k < l+2$, there is a unique choice for $\partial^{k-1}h_0^0/\partial y^{k-1}$, which forces $\tau^0(h) = O(y^{k+\varepsilon})$, in terms of previously determined data Q_k. We have $\tau^0(h) = O(y^{k+1})$, if $k+1 < l+2$.

These computations form the main part of the proof of

Proposition 5.1. *Suppose that* $f \in C^{l+2,\varepsilon}(\partial M', \partial\overline{M}')$, $0 \le l \le n_1 + 2n_2$, $0 < \varepsilon < 1$ *satisfies* $f_j^\gamma = 0$, $1 \le j \le n_1$, $\overline{n}_1 + 1 \le \gamma \le \overline{n}_1 + \overline{n}_2$, *and*

$$\sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} f_j^\gamma f_j^\gamma > 0.$$

Then there exists $h \in C^{l+2,\varepsilon}(M', \overline{M}')$, *assuming the boundary values* f, *with* $\|\tau(h)\| = O(y^{l+\varepsilon})$. *Moreover, the covariant derivatives of the tension satisfy* $\|\nabla^j\tau(h)\| = O(y^{l+\varepsilon})$, *for* $j \le l$.

Proof. If $l = 0$, this reduces to Proposition 4.2. The inductive scheme just given, for $2 \le k \le l+1$, $l \ge 1$, applies in local charts near $\partial M'$ to give $|\tau^\alpha(h)| = O(y^{l+2+\varepsilon})$, $\alpha \ge \overline{n}_1 + 1$; $|\tau^\alpha(h)| = O(y^{l+1+\varepsilon})$, $\alpha \ge 0$. Since the metric is given by (2.2), this means that $\|\tau(h)\| = O(y^{l+\varepsilon})$, in each chart near the boundary. However, a global solution was given in Proposition 4.2, when $l = 0$. At each stage of the inductive argument, in Cases 1, 2, 3, one uniquely determines the Taylor series modification of h. This uniqueness guarantees that the local solutions agree, to sufficiently high order in y, fitting together to give a global solution. The estimates for $\nabla^j\tau(h)$ follow from successive covariant differentiation, of the Taylor polynomial of $\tau(h)$ in y, using the orthonormal frame field $y\partial/\partial y$, yX_i, y^2Z_j. Since the coefficients, on the right-hand side of (2.4), are bounded, independent of y; and $h_j^\gamma = 0$, $0 \le j \le n_1$, $\overline{n}_1 + 1 \le \gamma \le \overline{n}_1 + \overline{n}_2$, at the boundary, $\|\nabla^j\tau(h)\|_{0,\varepsilon} = O(y^{l+\varepsilon})$, $j \le l$.

We may now apply the nonlinear heat equation to deform our higher order approximate solution to a harmonic map. The proof of Theorem 4.4 extends easily to give

Theorem 5.2. *Suppose that* $f \in C^{l+2,\varepsilon}(\partial M', \partial\overline{M}')$, $0 \le l < n_1 + 2n_2$, $0 < \varepsilon < 1$ *satisfies* $f_j^\gamma = 0$, $1 \le j \le n_1$, $\overline{n}_1+1 \le \gamma \le \overline{n}_1+\overline{n}_2$, *and* $\sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\overline{n}_1+1}^{\overline{n}_1+\overline{n}_2} f_j^\gamma f_j^\gamma > 0$. *Then there exists a harmonic map* u, *which assumes the boundary values* f *continuously, so that* $d(u, h) = O(y^{l+\overline{\varepsilon}})$, *any* $\overline{\varepsilon} < \varepsilon$, *where* h *is the map of Proposition* 5.1.

Proof. One follows the proof of Theorem 4.4, almost verbatim, using Proposition 5.1 rather than Proposition 4.2. The case $l = n_1 + 2n_2$ is excluded since then $s = l + \varepsilon > n_1 + 2n_2$, the superharmonic function ψ, of Lemma 4.3, only is available when $s \le n_1 + 2n_2$.

Suppose now that the range $\overline{M}$ is a hyperbolic space of constant curvature -1. The above construction of higher order approximation can then be modified to yield more attractive results. If the boundary data $f \in C^{1,\varepsilon}(\partial M', \partial\overline{M}')$, $0 < \varepsilon < 1$, satisfies the hypothesis of Proposition 4.6, then we showed that there exists an extension $h \in C^{1,\varepsilon}(M', \overline{M}') \cap C^2(M, \overline{M})$, with $\|\tau(h)\| = O(y^\varepsilon)$. For smoother boundary values $f \in C^{l+1,\varepsilon}(\partial M', \partial\overline{M}')$, we plan to modify h to achieve $\|\tau(h)\| = O(y^{l+\varepsilon})$, as long as $l \le n_1 + 2n_2$. It is already clear, from the proof of Proposition 4.6, that the h constructed there lies in $C^{l+1.\varepsilon}(M', \overline{M}')$, whenever $f \in C^{l+1,\varepsilon}(M', \overline{M}')$. The point is to improve the decay rate of the tension field.

To set up the induction, assume that $h \in C^{l+1,\varepsilon}$, $l \ge 1$, $0 < \varepsilon < 1$, has continuously assumed boundary values $f \in C^{l+1,\varepsilon}$. Assume that $h_0^0 > 0$; $h_0^\alpha = 0$, $1 \le \alpha \le \overline{n}_1 + \overline{n}_2$; $|\nabla_0^{l+2} h| = O(y^{\varepsilon-1})$, where ∇_0 denotes Euclidean derivatives in any local chart. If $k < l + 1$, assume that $\partial^{i-1} h_0^\alpha / \partial y^{i-1}$ are determined for $i \le k$, $\alpha \ge 0$. These modifications have been made to achieve $|\tau^\alpha(h)| = O(y^{k+1})$, all α. To start the induction, with $k = 1$, we invoke the proof of Proposition 4.6. Let Q_k denote a rational function of the already determine data.

Again, we use formulas (2.5a) and divide the discussion into the cases, depending upon the index α in $\tau^\alpha(h)$:

Case 1. $\alpha \ge 1$. Quoting from (2.5a) gives

$$\tau^\alpha(h) = g^{jj} h_{jj}^\alpha + (1 - n_1 - 2n_2) h_0^\alpha y - 2 g^{jj} h_j^0 h_j^\alpha \overline{y}^{-1}$$

with j summed from 0 to $n_1 + n_2$. Separating this into pieces corresponding to the splitting (2.2) of the metric, gives

$$\begin{aligned}\tau^\alpha(h) = {} & y^2 h_{00}^\alpha + y^2 \sum_{j=1}^{n_1} h_{jj}^\alpha + y^4 \sum_{j=n_1+1}^{n_1+n_2} h_{jj}^\alpha + (1 - n_1 - 2n_2) h_0^\alpha y \\ & - 2y^2 h_0^0 h_0^\alpha \overline{y}^{-1} - 2y^2 \sum_{j=1}^{n_1} h_j^0 h_j^\alpha \overline{y}^{-1} - 2y^4 \sum_{j=n_1+1}^{n_1+n_2} h_j^0 h_j^\alpha \overline{y}^{-1}.\end{aligned}$$

We now identify the terms Q_k already fixed:

$$\tau^\alpha(h) = y^2 h_{00}^\alpha + (1 - n_1 - 2n_2) h_0^\alpha y - 2y^2 h_0^0 h_0^\alpha \overline{y}^{-1} + Q_k y^{k+1} + O(y^{k+1+\varepsilon}).$$

Here, we used the fact that $h_j^0 = 0$, at the boundary, $j \ge 1$. Using the Taylor polynomial of the significant terms:

$$k!\tau^\alpha(h) = (k - n_1 - 2n_2 - 1) \frac{\partial^k h_0^\alpha}{\partial y^k} y^{k+1} + Q_k y^{k+1} + O(y^{k+1+\varepsilon}).$$

If $k + 1 < l + 1$, then the remainder is of order $O(y^{k+2})$. Since $k < l + 1$ and $l \le n_1 + 2n_2$, the derivative $\partial^k h_0^\alpha / \partial y^k$ is uniquely determined, in terms of the previously known data Q_k, to give $\tau^\alpha(h) = O(y^{k+1+\varepsilon})$, and in fact the better condition $\tau^\alpha(h) = O(y^{k+2})$, as long as $k + 1 < l + 1$.

Case 2. $\alpha = 0$. Using (2.5a) again,

$$\tau^0(h) = g^{jj}h^0_{jj} + (1 - n_1 - 2n_2)h^0_0 y - g^{jj}h^0_j h^0_j \overline{y}^{-1} + g^{jj}\sum_{\gamma=1}^{\overline{n}_1} h^\gamma_j h^\gamma_j \overline{y}^{-1}$$

where $0 \le j \le n_1 + n_2$ is summed. Separating into pieces corresponding to the splitting (2.2) of the metric yields

$$\begin{aligned}\tau^0(h) = y^2 h^0_{00} &+ \sum_{j=1}^{n_1} y^2 h^0_{jj} + \sum_{j=n_1+1}^{n_1+n_2} y^4 h^0_{jj} + (1 - n_1 - 2n_2)h^0_0 y \\ &- y^2(h^0_0)^2\overline{y}^{-1} - y^2\sum_{j=1}^{n_1} h^0_j h^0_j \overline{y}^{-1} - y^4 \sum_{j=n_1+1}^{n_1+n_2} h^0_j h^0_j \overline{y}^{-1} \\ &+ y^2 \sum_{\gamma=1}^{\overline{n}_1} h^\gamma_0 h^\gamma_0 \overline{y}^{-1} + \sum_{j=1}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} y^2 h^\gamma_j h^\gamma_j \overline{y}^{-1} \\ &+ y^4 \sum_{j=n_1+1}^{n_1+n_2}\sum_{\gamma=1}^{\overline{n}_1} h^\gamma_j h^\gamma_j \overline{y}^{-1}.\end{aligned}$$

Identifying terms of type Q_k, which are already determined:

$$\begin{aligned}\tau^0(h) = y^2 h^0_{00} &+ (1 - n_1 - 2n_2)h^0_0 y - y^2(h^0_0)^2\overline{y}^{-1} \\ &+ y^2\overline{y}^{-1}\sum_{j=1}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} h^\gamma_j h^\gamma_j + Q_k y^{k+1} + O(y^{k+1+\varepsilon}).\end{aligned}$$

Here, we used the hypotheses that $h^0_j = 0$, $j \ge 1$, and $h^\alpha_0 = 0$, $\alpha \ge 1$, along the boundary.

Expanding the relevant terms in Taylor series yields

$$\begin{aligned}&(k+1)!\tau^0(h) \\ &= \left[1 + (k+1)(k - n_1 - 2n_2 - 1) - \left(\frac{1}{h^0_0}\right)^2 \sum_{j=1}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} h^\gamma_j h^\gamma_j\right] \frac{\partial^k h^0_0}{\partial y^k} y^{k+1} \\ &\quad + Q_k y^{k+1} + O(y^{k+\varepsilon+1}).\end{aligned}$$

If $k+1 < l+1$, the remainder is $O(y^{k+2})$. Since $k < l+1$ and $l \le n_1 + 2n_2$, the derivative $\partial^k h^0_0/\partial y^k$ is uniquely determined, in terms of already known data Q_k, to give $\tau^0(h) = O(y^{k+1+\varepsilon})$, and actually $\tau^0(h) = O(y^{k+2})$, when $k+1 < l+1$.

Arguing as in the proof of Proposition 5.1, we employ these calculations to deduce an extension of Proposition 4.6.

Proposition 5.3. *Assume that* $f \in C^{l+1,\varepsilon}(\partial M', \partial\overline{M}')$, $0 < \varepsilon < 1$, *satisfies* $\sum_{j=1}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} f^\gamma_j f^\gamma_j > 0$. *Then there exists* $h \in C^{l+1,\varepsilon}(M', \overline{M}') \cap C^2(M, \overline{M})$, $l \ge 0$, *assuming the boundary values* f *continuously, with* $\|\tau(h)\| = O(y^{l+\varepsilon})$, *as long as* $l \le n_1 + 2n_2$. *Moreover, the covariant derivatives of the tension satisfy* $\|\nabla^j\tau(h)\|_{0,\varepsilon} = O(y^{l+\varepsilon})$, *for* $j \le l$.

Applying the nonlinear heat equation, with initial data h, gives the analogue of Theorem 5.2.

Theorem 5.4. *Suppose that $\overline{M}$ is a hyperbolic space of constant negative curvature -1. Assume that $f \in C^{l+1,\varepsilon}(\partial M', \partial\overline{M}')$, $0 < \varepsilon < 1$, $0 \le l < n_1 + 2n_2$, satisfies $\sum_{j=1}^{n_1}\sum_{\gamma=1}^{\overline{n}_1} f_j^\gamma f_j^\gamma > 0$. Then there exists a harmonic map u, assuming the boundary values f continuously, so that $d(u, h) = O(y^{l+\overline{\varepsilon}})$, and $\overline{\varepsilon} < \varepsilon$, where h is the map of Proposition* 5.3.

6. Boundary regularity

Assume that M and $\overline{M}$ are globally symmetric spaces of noncompact type and rank one. Given $f \in C^{l+2,\varepsilon}(\partial M', \partial\overline{M}')$, satisfying the hypotheses of Proposition 5.1, we extended f to an asymptotically harmonic map $h \in C^{l+2,\varepsilon}(M', \overline{M}')$, with $\|\nabla^j \tau(h)\|_{0,\varepsilon} = O(y^{l+\varepsilon})$, $j \le l$. The nonlinear heat equation was then employed, in the proof of Theorem 5.2, to deform h to a harmonic map u, with $d(u, h) = O(y^{l+\overline{\varepsilon}})$, any $\overline{\varepsilon} < \varepsilon$, so that u is asymptotically close to h, measured in the hyperbolic distance d. Clearly, u assumes the boundary values f continuously. It is natural to expect that u will also inherit the boundary regularity of h. The key to our approach, to this issue, is the following:

Lemma 6.1. *Let h and u be the maps constructed in Proposition* 5.1 *and Theorem* 5.2, *respectively. If $p \in M$ is Euclidean distance y from the boundary, then consider the representation of h and u, relative to Riemannian normal coordinates, on unit balls $B(p, 1)$ and $B(h(p), 1)$. In Hölder norm, relative to these normal coordinates, $\|u - h\|_{l+2,\varepsilon} = O(y^{l+\overline{\varepsilon}})$, any $\overline{\varepsilon} < \varepsilon$.*

Proof. Since the metrics on M and $\overline{M}$ admit transitive groups of isometries, the Christoffel symbols, and their derivatives to any order, are bounded on $B(p, 1)$ and $B(h(p), 1)$, independent of p. This may be seen by composing with isometries which move p, $h(p)$ back to fixed reference points. The usual coordinates representation of the tension field gives

$$\Delta u^\alpha + (\Gamma^\alpha_{\beta\gamma} \circ u)\frac{\partial u^\beta}{\partial x_i}\frac{\partial u^\gamma}{\partial x_j} g^{ij} = 0,$$

$$\Delta h^\alpha + (\Gamma^\alpha_{\beta\gamma} \circ h)\frac{\partial h^\beta}{\partial x_i}\frac{\partial h^\gamma}{\partial x_j} g^{ij} = \tau(h).$$

Here Δ is the Laplace operator of the Riemannian metric on M. Latin indices refer to M and Greek indices refer to $\overline{M}$.

The hypotheses of Proposition 5.1, and the method of construction of h and u, show that both of these maps have bounded energy density. By Schauder theory [3], u and h are each bounded in $C^{1,\varepsilon}$. Since the coefficients of the tension field equation are now C^ε bounded, u and h are bounded in $C^{2,\varepsilon}$ norm. A standard iteration argument then bounds u and v in $C^{l+2,\varepsilon}$ norm.

We let $w = u - h$. Taking the difference of the tension field equations gives

$$\Delta w^\alpha + (\Gamma^\alpha_{\beta\gamma} \circ u)\frac{\partial w^\beta}{\partial x_i}\frac{\partial u^\gamma}{\partial x_j} g^{ij} + (\Gamma^\alpha_{\beta\gamma} \circ u)\frac{\partial h^\beta}{\partial x_i}\frac{\partial w^\gamma}{\partial x_j} g^{ij}$$
$$= (-\Gamma^\alpha_{\beta\gamma} \circ u + \Gamma^\alpha_{\beta\gamma} \circ h)\frac{\partial h^\beta}{\partial x_i}\frac{\partial h^\gamma}{\partial x_j} g^{ij} - \tau(h).$$

This is a linear equation for w with $C^{l+1,\varepsilon}$ bounded coefficients. Since w and the inhomogeneous term are of order $O(y^{l+\bar{\varepsilon}})$, in C^0 norm, Schauder theory shows that w is $O(y^{l+\bar{\varepsilon}})$ in $C^{1,\varepsilon}$ norm. The right-hand side is now bounded, in C^{ε} norm, by $O(y^{l+\bar{\varepsilon}})$. Schauder theory shows that w is $O(y^{l+\bar{\varepsilon}})$ in $C^{2,\varepsilon}$ norm. Iteration yields the desired bound $\|u-h\|_{l+2,\varepsilon} = O(y^{l+\bar{\varepsilon}})$.

We apply this lemma to deduce our main result concerning boundary regularity.

Theorem 6.2. *Suppose that* $f \in C^{l+2,\varepsilon}(\partial M', \partial \overline{M}')$, $0 \le l < n_1 + 2n_2$, $0 < \varepsilon < 1$ *satisfies* $f_j^{\gamma} = 0$, $1 \le j \le n_1$, $\bar{n}_1 + 1 \le \gamma \le \bar{n}_1 + \bar{n}_2$, *and* $\sum_{j=n_1+1}^{n_1+n_2} \sum_{\gamma=\bar{n}_1+1}^{\bar{n}_1+\bar{n}_2} f_j^{\gamma} f_j^{\gamma} > 0$. *Then there exists a harmonic map* u, *with boundary values* f, *and* $u \in C^{k+1,\bar{\varepsilon}}(M', \overline{M}')$, *for* $-2 \le 2k < l-1$, *any* $\bar{\varepsilon} < \varepsilon$.

Proof. Let u be the harmonic map constructed in Theorem 5.2 and h the asymptotically harmonic map of Proposition 5.1. In local Euclidean charts, near the boundaries of the compactifications, we have $|du - dh| = O(y^{l-1+\bar{\varepsilon}})$, by Lemma 6.1. The factor -1 enters because the metric (2.2) is not isotropic.

For higher derivatives, we consider the orthonormal frame field $y\partial/\partial y$, yX_i, $y^2 Z_j$ on M, with its complete Riemannian metric, and the corresponding frame field on the image $\overline{M}$. Formula (2.4), for covariant derivatives in the frame field, has constant coefficients on the right-hand side. Therefore, it is comparable to the Riemannian normal coordinate frame fields used in Lemma 6.1. It follows, by induction in k, that $|V_1 V_2 \cdots V_k(du - dh)| = O(y^{l+\bar{\varepsilon}})$, where $du - dh$ is realized as a matrix in the Riemannian orthonormal frame fields, and each V_i belongs to our chosen orthonormal frame field.

We now convert to the Euclidean reference frame $\partial/\partial y$, X_i, Z_j. If each $W_s \in \{\partial/\partial y, X_i, Z_j\}$, then $|W_1 W_2 \cdots W_k(du - dh)| = O(y^{l-1+\bar{\varepsilon}-2k})$, where $du - dh$ is realized as a matrix in the Euclidean frame. The factor 2 enters, in the exponent, because of differentiations is the directions Z_j, which correspond to $y^2 Z_j$ in the Riemannian orthonormal frame field. As long as, $2k < l-1$, we see that u and h agree, along the boundary, up to order $k+1$.

Suppose now that the range $\overline{M}$ is a hyperbolic space of constant negative curvature -1. In this case, we apply similar arguments, starting with Theorem 5.4, to deduce

Theorem 6.3. *Assume that* $\overline{M}$ *is of constant negative curvature. Let* $f \in C^{l+1,\varepsilon}(\partial M', \partial \overline{M}')$, $0 < \varepsilon < 1$, $0 \le l < n_1 + 2n_2$, *satisfy the condition*

$$\sum_{j=1}^{n_1} \sum_{\gamma=1}^{\bar{n}_1} f_j^{\gamma} f_j^{\gamma} > 0.$$

Then there exists a harmonic map u, *assuming the boundary values* f, *and moreover* $u \in C^{k+1,\bar{\varepsilon}}(M', \overline{M}')$, *for* $-2 \le 2k < l-1$, *for* $\bar{\varepsilon} < \varepsilon$.

If both M and $\overline{M}$ have constant negative curvature, the same argument gives a different proof of the following result from [8].

Theorem 6.4. *Suppose that the hypotheses of Theorem* 6.3 *are satisfied and in addition that* M *has constant negative curvature. Then* $u \in C^{l+1,\bar{\varepsilon}}(M', \overline{M}')$, *for any* $\bar{\varepsilon} < \varepsilon$.

Proof. Both the metrics on M and $\overline{M}$ are conformal to the Euclidean metrics in our local charts, by comparable factors, since $h_0^0 > 0$. Thus $|du - dh| = O(y^{l+\bar{\varepsilon}})$, in the Euclidean sense. For the higher derivatives we use $V_s \in \{y\partial/\partial y, yX_i\}$ and $W_s \in \{\partial/\partial y, X_i\}$. The factor 2, from the directions Z_j, no longer appears. Thus $|W_1 W_2 \cdots W_k(du - dh)| = O(y^{l+\bar{\varepsilon}-k})$, allowing us to choose $k \leq l$.

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